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Groundwater Modelling

From a simple water balance to the numerical solution of the groundwater flow equation

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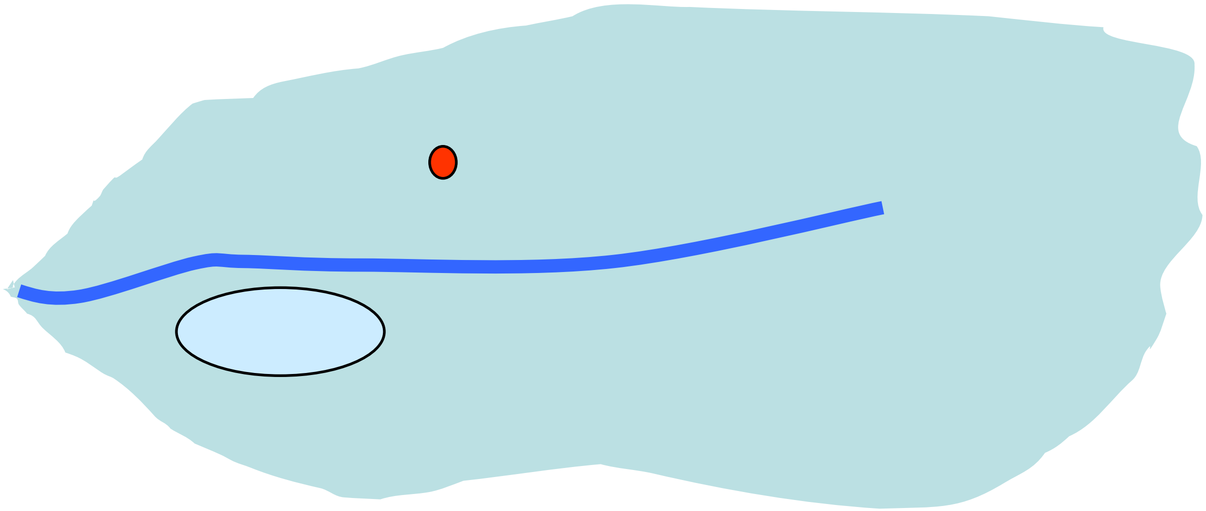
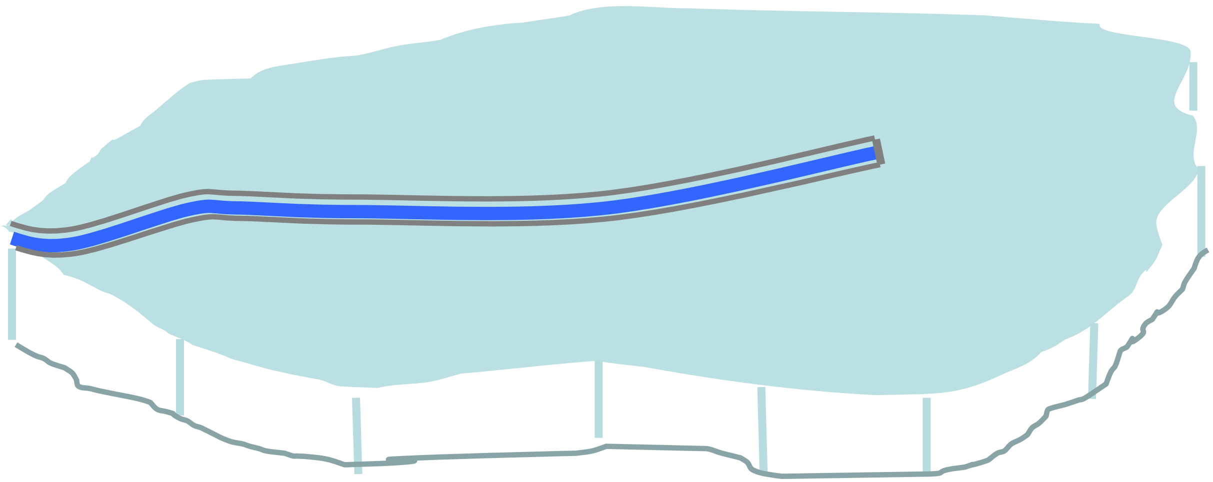
Water balance of an aquifer



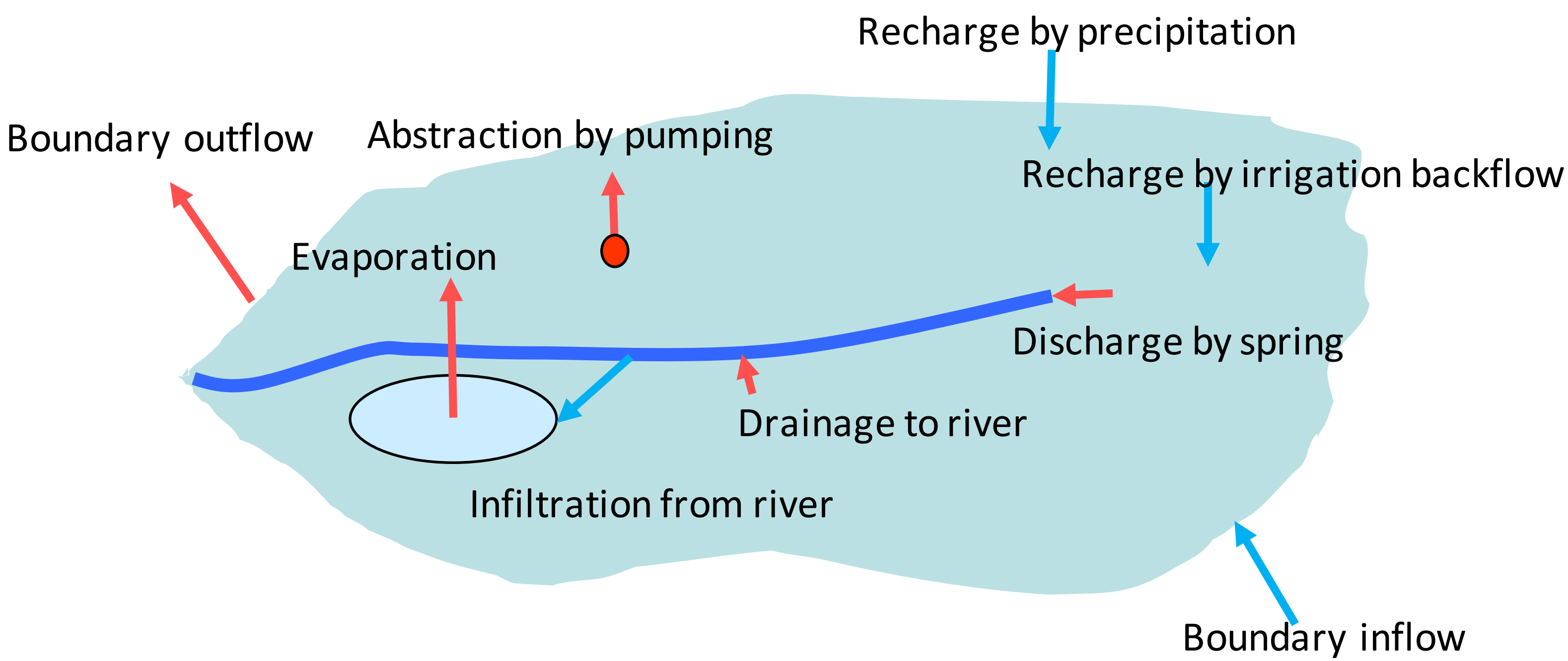
- Choose boundary for balance
- Preferably natural boundaries such as end of aquifer, river, stream line, water divide, line of constant head

Choice of boundary for balance



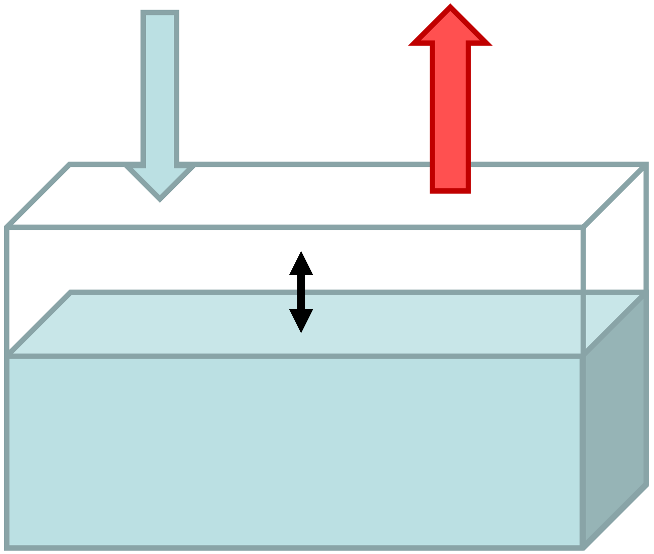
- Take only aquifer, ending at river bed
- Upper boundary: top of confined aquifer or water table of phreatic aquifer
- Lower boundary: aquifer bottom

Water balance of an aquifer



- Identify Inflows to region and outflows from region
- Sign convention: into aquifer is positive (blue), out is negative (red)

Schematic water balance of aquifer



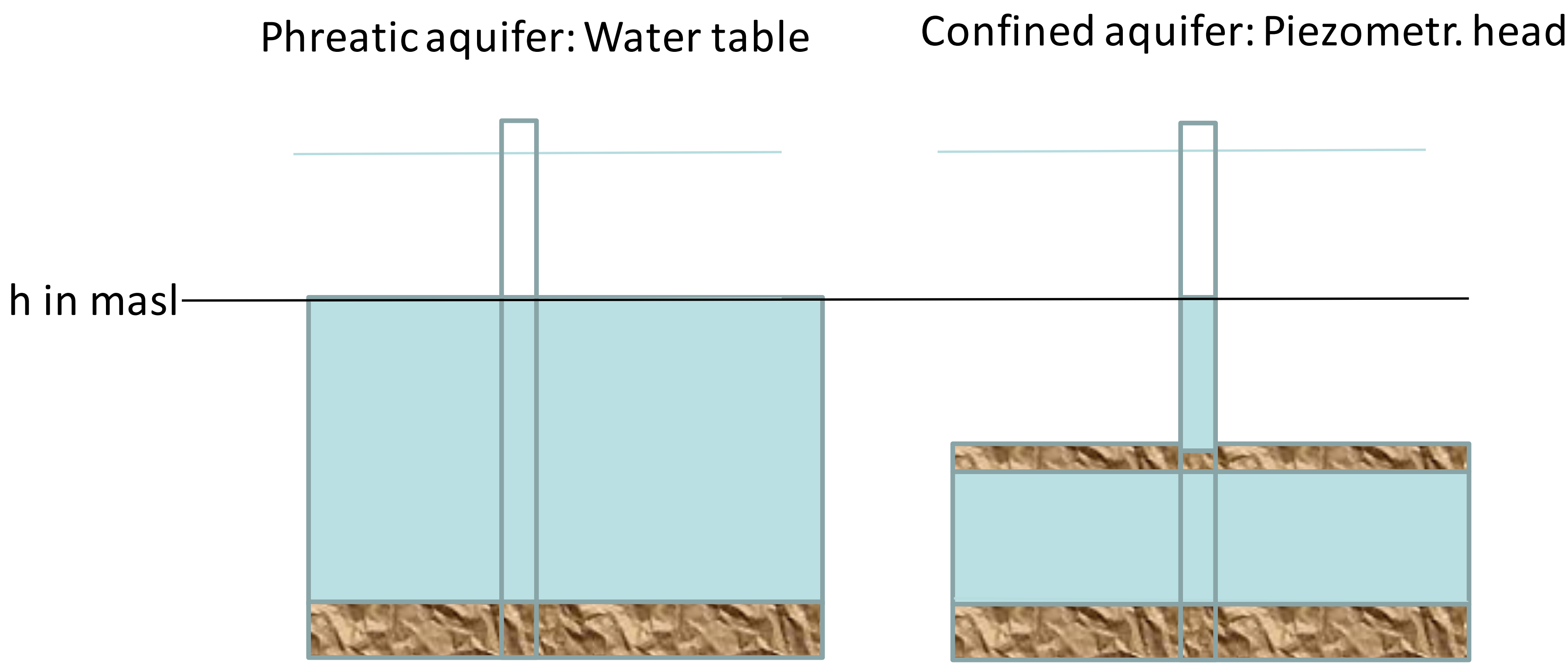
also called:
Box Model

- Sum of all fluxes = ∑In – ∑Out = Storage (or destorage)
- In = Out: steady state
- In > Out: water table (head) rises, storage
- In < Out: water table (head) declines, destorage

Box model should always be our first step!

Head/Water table

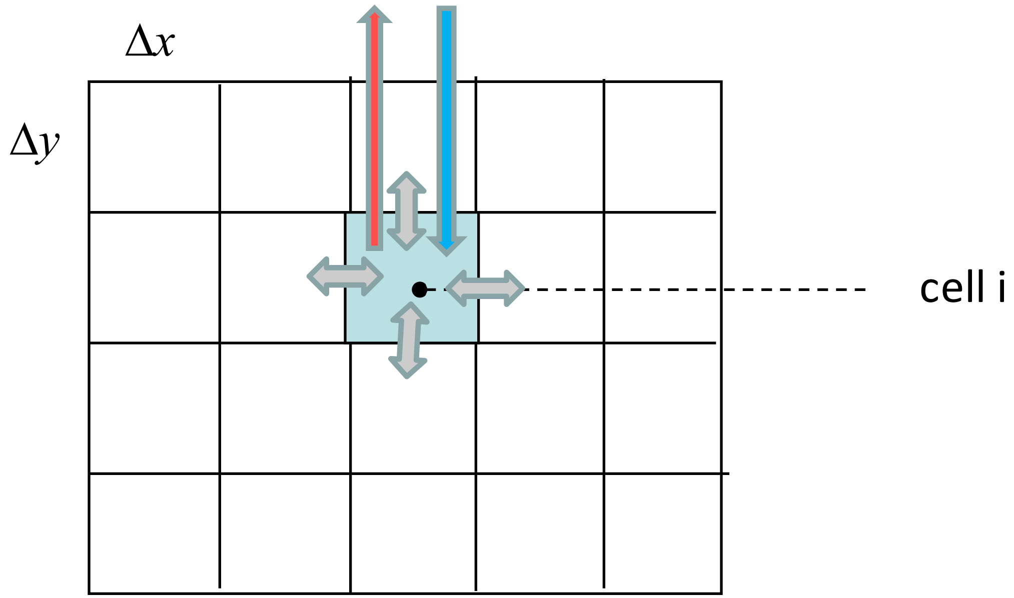
- What we see in a piezometer



Why a numerical model?

- A box model does not resolve local details like cones of depression etc.
- It cannot easily be compared to local measurements of groundwater tables (heads) and groundwater head contours
- Internal positive and negative fluxes may be big and in their respective location damaging while they compensate in the box model

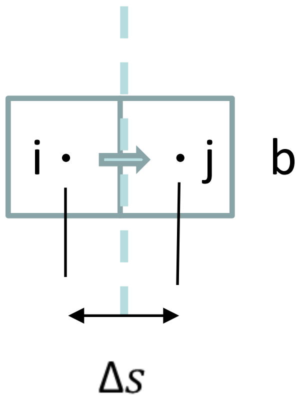
What is a numerical groundwater model?



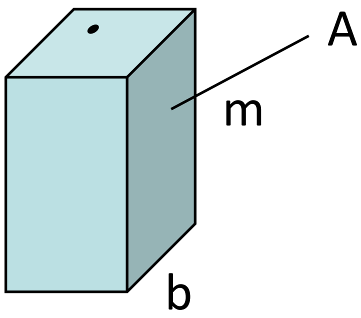
- A discretized form of a water balance to resolve more local details
- Divide the aquifer into a set of cells i= 1,…,N
- Set up a balance equation for each cell
- Use Darcy's law to express horizontal exchange of water between cells
- Obtain N balance equations with N unknown heads
- Apply boundary conditions at the boundary to close the equation system

What is Darcy's law

- Specific flux $v = KI$
  - with K hydraulic conductivity (m/d or m/s)
  - I hydraulic gradient $I = -\frac{h_j - h_i}{\Delta s}$
  - v Darcy velocity



- Flux over side of cell $Q = Av = bmKI = bTI$
  - with b width
  - m thickness
  - T transmissivity (m^2/d or m^2/s)



Boundary conditions

Flux boundary

- Impervious (Flux = 0) e.g. water divide, rock
- Inflow/Outflow (Flux = +/-) Inflow at edge from slopes

Prescribed head boundary

- h given (e.g. at river, drain)

Third type boundary: Mixture of the two

- Outside head connected via resistance (e.g. river with colmatation, clogged drain)

Simple example

Steady state flow in aquifer with 6 cells

$T_1 = T_3 = T_5 = T_6 = 0.05$ m^2 s^{-1} $T_2 = T_4 = 0.01$ m^2 s^{-1}

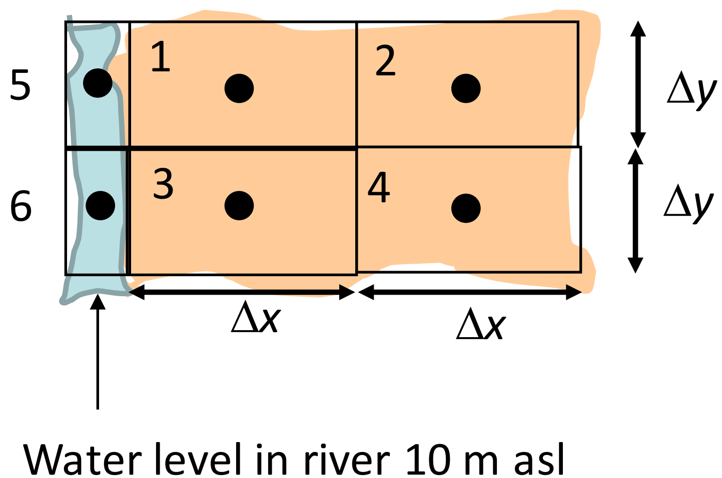
$h_5 = h_6 = 50$ m

$Q_1 = -0.1$ m^3 s^{-1} $Q_4 = -0.6$ m^3 s^{-1}

$\Delta x = 6000$ m $\Delta y = 3000$ m

Recharge by precipitation $N = 10^{-8}$ m s^{-1}

River cells 5-6: Assume here negligible width of river cell of 10 m << 6000 m



Boundary conditions??

head given at 5,6, impervious otherwise

Simple example: Equation system

Water balance for cells 1-4

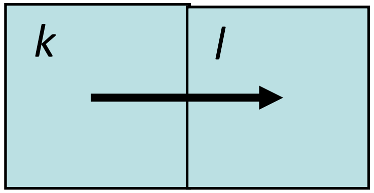
$$Q_{51}+Q_{21}+Q_{31}+Q_1+N\ \ x\ \ y=0$$
$$Q_{12}+Q_{42}+N\ \ x\ \ y=0$$
$$Q_{63}+Q_{13}+Q_{43}+N\ \ x\ \ y=0$$
$$Q_{34}+Q_{24}+Q_4+N\ \ x\ \ y=0$$

Insert Darcy:

$$\frac{h_5\quad h_1}{x/2}\ \ yT_{51}+\frac{h_2\quad h_1}{x}\ \ yT_{21}+\frac{h_3\quad h_1}{y}\ \ xT_{31}+Q_1+N\ \ x\ \ y=0$$
$$\frac{h_1\quad h_2}{x}\ \ yT_{12}+\frac{h_4\quad h_2}{y}\ \ xT_{42}+N\ \ x\ \ y=0$$
$$\frac{h_6\quad h_3}{x/2}\ \ yT_{63}+\frac{h_1\quad h_3}{y}\ \ xT_{13}+\frac{h_4\quad h_3}{x}\ \ yT_{43}+N\ \ x\ \ y=0$$
$$\frac{h_3\quad h_4}{x}\ \ yT_{34}+\frac{h_2\quad h_4}{y}\ \ xT_{24}+Q_4+N\ \ x\ \ y=0$$
$$h_5=10\mathrm{m}\qquad h_6=10\mathrm{m}$$

Transmissivity between cells

Evaluate mean transmissivity:



How to average?

Harmonic mean:

$$T_{kl} = \frac{2T_k T_l}{T_k + T_l}$$

Arithmetic mean:

$$T_{kl} = \frac{1}{2}\left(T_k + T_l\right)$$

Harmonic mean is better!

Simple example: Solution

Equation system:

$$
\begin{aligned}
a_{11}h_1 + a_{12}h_2 + a_{13}h_3 \qquad\qquad &= b_1 \\
a_{21}h_1 + a_{22}h_2 \qquad\qquad + a_{24}h_4 &= b_2 \\
a_{31}h_1 \qquad\qquad + a_{33}h_3 + a_{34}h_4 &= b_3 \\
a_{42}h_2 + a_{43}h_3 + a_{44}h_4 &= b_4
\end{aligned}
$$

Gauss-Seidel approach (iterative):

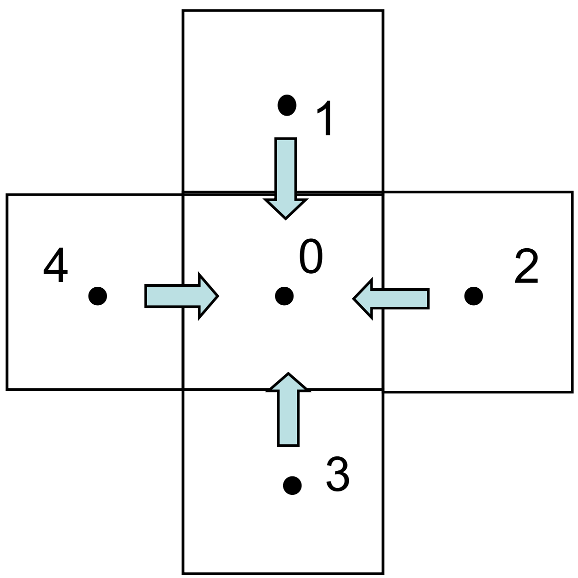
- Solve for diagonal element.
- Use most recent estimate.
- Stop iteration if increment smaller than given ε.
- Not suited for large systems

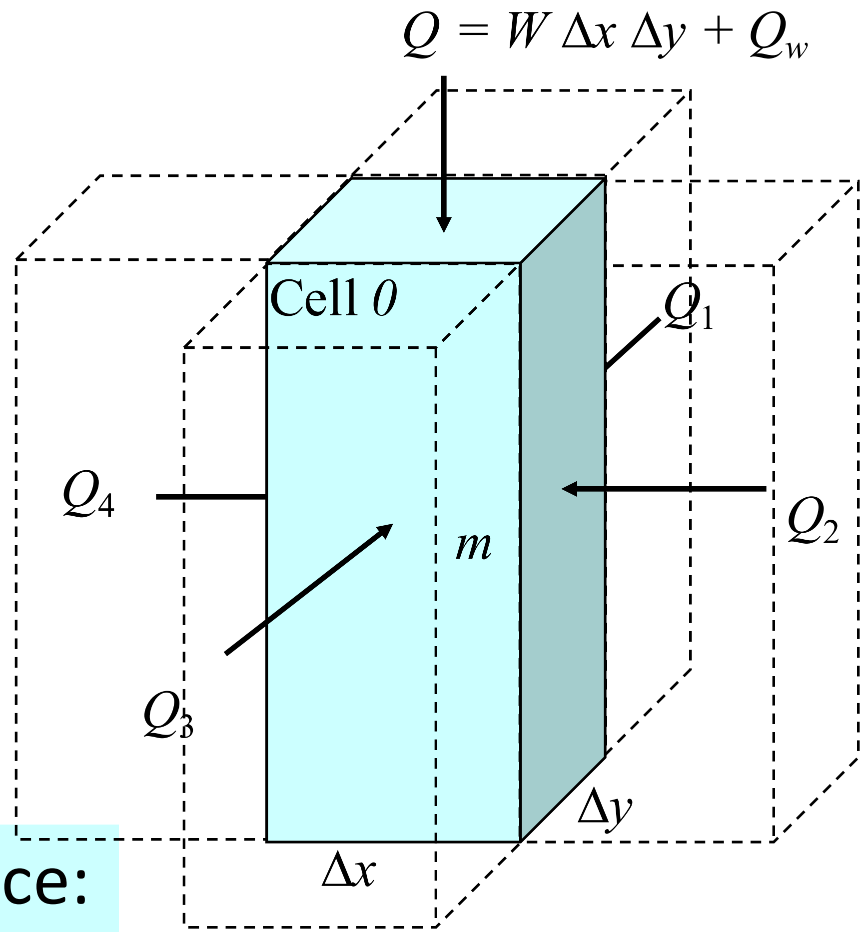
$$
\begin{aligned}
h_1^{new} &= \left(b_1 \quad a_{12}h_2^{old} \quad a_{13}h_3^{old}\right)/a_{11} \\
h_2^{new} &= \left(b_2 \quad a_{21}h_1^{new} \quad a_{24}h_4^{old}\right)/a_{22} \\
h_3^{new} &= \left(b_3 \quad a_{31}h_1^{new} \quad a_{34}h_4^{old}\right)/a_{33} \\
h_4^{new} &= \left(b_4 \quad a_{42}h_2^{new} \quad a_{43}h_3^{new}\right)/a_{44}
\end{aligned}
$$

Procedure

- Formulate **water balance for each cell**.
- Formulate water flux through cell boundaries using Darcy's law and head at cell centres. Add other fluxes.
- Formulate temporal change in water volume.
- Take into account boundary conditions.
- Solve resulting equation system for heads at cell centres.
- In time varying case add storage term to close the balance

Water balance in considered cell





Water volume balance:

$$t\left(Q_1 + Q_2 + Q_3 + Q_4 + Q\right) = \left(h_0(t + \quad t) \quad h_0(t)\right)S \;\; x \;\; y$$

Storage coefficient

- Specific storage coefficient S_0 is the amount of water stored additionally in a colum of 1 m^2 area and 1 m height per increase of head by 1 m (unit 1/m).
- Storage coefficient $S = S_0 m$ where m is the thickness of the aqujifer (m = top-bottom) (unit -)
- In a phreatic aquifer S is equal to the specific yield n (unit -)

Water fluxes to considered cell

Fluxes according to Darcy‘s law:

$$Q_1 = \quad xT_{10} \frac{h_1(t') \quad h_0(t')}{y} \qquad\qquad Q_2 = \quad yT_{20} \frac{h_2(t') \quad h_0(t')}{x}$$

$$Q_3 = \quad xT_{30} \frac{h_3(t') \quad h_0(t')}{y} \qquad\qquad Q_4 = \quad yT_{40} \frac{h_4(t') \quad h_0(t')}{x}$$

T_{kl} Transmissivity between nodes k und l

Time for the evaluation of fluxes: $t' \quad [t, t + \quad t]$

Which time should be chosen? Use t’= t+Δt

Water balance in considered cell

Water balance:

$$S\frac{\left(h_0(t+\;\;t)\quad h_0(t)\right)}{t}=$$

$$T_{10}\frac{h_1(t')\quad h_0(t')}{y^2}+T_{20}\frac{h_2(t')\quad h_0(t')}{x^2}+$$

$$T_{30}\frac{h_3(t')\quad h_0(t')}{y^2}+T_{40}\frac{h_4(t')\quad h_0(t')}{x^2}+q_0$$

With recharge rate *N* and pumping rate Q_w:

$$q_0=W=N+Q_w/(\;\;x\;\;y)$$

For steady state flow remove transient term

Solution of equation system

Direct solvers:

Gauss elimination (slow, computational effort prop. to N^3).

Special solvers for banded matrix structure.

Iterative solvers:

Gauss-Seidel

Preconditioned Conjugate Gradients solvers

Multi-grid techniques

Additional remarks

- Cell size Δx and Δy need not be constant.
- There is also the alternative of unstructured grids (integrated finite differences)
- Ratio of adjacent cell sizes should be ≤ 2.
- Boundary conditions can be transient.